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Exact Expressions for Curved Characteristics Behind Strong Blast Waves

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I. Introduction

Blast waves are produced in gaseous media due to the sudden deposition of large amounts of energy in relatively small regions. The propagation of a point-source blast wave into an ideal gas, whose initial pressure is assumed to be negligibly low, is known to be self-similar. This property was first deduced by Taylor[1] using dimensional arguments. A brief derivation of this result based on invariant theorems of continuous group of transformations is given in the appendix. Closed form solutions describing the flow variables in the nonisentropic region behind such a blast wave in $n(= 1, 2, 3)$ dimensions were obtained independently by von Neumann[2] and Sedov[3]. The reflection of strong blast waves had been discussed in an earlier paper[4] by the authors.

Since the self-similar solution is a solution with stratified entropy - in fact, the only exact solution of this type known and therefore interesting pedagogically - there are three sets of characteristic curves.* The purpose of this paper is to report the interesting result that these three families of curved characteristics also can be represented in closed form, and remarkably still in terms of elementary functions.

*See, e.g., Courant & Friedrichs[5].

As far as the authors know, these expressions represent the only known closed form solutions of three sets of curved characteristics in a non-isentropic flow region.

II. Self-Similar Solution

Using the notations of Chang & Laporte[4], the flow variables behind a self-similar blast wave may be expressed in dimensionless forms as follows:

$$f = u/u' = A_1 \hat{u} y \quad ,$$

$$g = p/p' = (A_1 \hat{u})^{A_2 n} [A_3 (1 - \hat{u}/A_2)]^{A_4}$$

$$\cdot [A_1 (1 - A_5 \hat{u})/(A_1 - A_5)]^{2A_6} \quad ,$$

$$h = \rho/\rho' = [A_3 (\gamma\hat{u}/A_2 - 1]^{A_7} [A_3 (1 - \hat{u}/A_2)]^{A_4 - 1}$$

$$\cdot [A_1 (1 - A_5 \hat{u})/(A_1 - A_5)]^{2A_9} \quad . \qquad (1)$$

In these equations,

$$y = r/R = (A_1 \hat{u})^{-A_2} [A_3 (\gamma\hat{u}/A_2 - 1)]^{A_8}$$

$$\cdot [A_1 (1 - A_5 \hat{u})/(A_1 - A_5)]^{A_6 - A_9} \qquad (2)$$

is the similarity parameter, and (u, p, ρ) are the velocity, pressure, and density in the nonisentropic flow region behind the blast wave, respectively. Furthermore,

$$\hat{u} = ut/r \tag{3}$$

is a dimensionless velocity, t is the elapsed time, r and R respectively are the radial distance and shock radius measured from the point of explosion. The constants A_1, A_2, ..., A_9 are defined in terms of the adiabatic index γ and n as follows:

$$A_1 = \frac{(2 + n)(\gamma + 1)}{4} \quad , \quad A_2 = \frac{2}{2 + n} \quad , \quad A_3 = \frac{\gamma + 1}{\gamma - 1} \quad ,$$

$$A_4 = \frac{\gamma}{\gamma - 2} \quad , \quad A_5 = \frac{2 + n(\gamma - 1)}{2} \quad ,$$

$$A_6 = \frac{n(2\gamma + n - 2)}{(2 + n)(n\gamma - n + 2)} + \frac{(2 + n)\,\gamma\,(\gamma - 1)}{2(2 - \gamma)(n\gamma - n + 2)} \quad ,$$

$$A_7 = \frac{n}{2\gamma + n - 2} \quad , \quad A_8 = \frac{\gamma - 1}{2\gamma + n - 2} \quad ,$$

$$A_9 = \frac{n}{n\gamma - n + 2} + \frac{(2 + n)^2\,\gamma\,(\gamma - 1)}{2(2 - \gamma)(n\gamma - n + 2)(2\gamma + n - 2)} \quad , \tag{4}$$

where n = 1 for a planar wave, = 2 for a cylindrical wave, and = 3 for a spherical wave. The flow velocity u', fluid pressure p', the fluid density ρ', just behind the blast wave at r = R (or y = 1) are expressible in terms of the instantaneous shock speed U, and the fluid density ρ_o ahead of the shock wave (i.e., the initial density) by the familiar strong shock formulae as follows:

$$u' = \frac{2}{\gamma + 1} U \quad , \quad p' = \frac{2}{\gamma + 1} \rho_o U^2 \quad , \quad \rho' = \frac{\gamma + 1}{\gamma - 1} \rho_o \quad . \tag{5}$$

Following an argument due to Taylor[1] or the group-theoretical discussion given in the appendix, the shock radius R may be expressed as a function of the elapsed time t as follows:

$$R = K_n \, (E_o/\rho_o)^{1/(2 + n)} \, t^{2/(2 + n)} \quad , \tag{6}$$

where E_o is the energy released per unit area for a planar wave, per unit length for a cylindrical wave, and the total energy released for a spherical wave, and K_n is a constant determined by the energy integral

$$E_o = \frac{\pi^{n/2}}{\Gamma(n/2)} \int_o^R \rho r^{n - 1} \left(u^2 + \frac{2}{\gamma - 1} \frac{p}{\rho}\right) dr \quad . \tag{7}$$

From Eq. (6), the shock speed U can be calculated. The expressions for f, g, and h, given in Eqs. (1), can be regarded as functions of the similarity parameter y which has the convenient range of $0 \leq y \leq 1$.

III. Particle Lines

It is known that the basic equations governing the nonisentropic flow behind a propagating blast wave admit three distinct characteristic directions in the r-t plane given by

$$dr/dt = u, \, u \pm a \quad , \tag{8}$$

where $a = (\gamma p/\rho)^{1/2}$ is the local adiabatic speed of sound. The first characteristic direction given by Eq. (8) coincides with the local particle velocity and the family of characteristics are therefore particle lines. This direction corresponds to the speed of propagation of entropy disturbances. The other two characteristic directions of Eq. (8) correspond to the local speeds of propagation of pressure, density, or velocity disturbances.

Consider first the (u)-characteristics or particle lines. According to its definition and the self-similar solution, Eqs. (1)-(7), it may be easily deduced that along a (u)-characteristic,

$$dr/dt = 2A_1 U \hat{u} y(\hat{u})/(\gamma + 1) \quad , \tag{9}$$

where $y(\hat{u})$ is given by Eq. (2). The shock speed U can be evaluated from Eq. (6) as follows:

$$\begin{aligned} U &= [2K_n/(2 + n)] \, (E_o/\rho_o)^{1/(2 + n)} \, t^{-n/(2 + n)} \\ &= [2/(2 + n)] \, (R/t) \quad . \end{aligned} \tag{10}$$

But, from the definition of the similarity parameter, Eq. (2), and the expression for the shock radius, Eq. (6), it can be shown that, in general,

$$\begin{aligned} dr/dt = K_n \, (E_o/\rho_o)^{1/(2 + n)} \, t^{-n/(2 + n)} \\ \cdot [2y/(2 + n) + t \, dy/dt] \quad . \end{aligned} \tag{11}$$

Therefore, from Eqs. (2) and (9)-(11), it is found that,

$$d \log t = \frac{d \log y}{d\hat{u}} \cdot \frac{d\hat{u}}{\hat{u} - 2/(2 + n)} \quad , \tag{12}$$

along a particle line. But, from Eq. (2),

$$\frac{d \log y}{d\hat{u}} = \frac{(\gamma - 1)\, \hat{u}^2/2 + [\hat{u} - 2/(2 + n)][\hat{u} - 2/\gamma(2 + n)]}{\hat{u}[2/\gamma(2 + n) - \hat{u}][(2 - n + n\gamma)\, \hat{u}/2 - 1]} \quad . \tag{13}$$

Therefore,

$$d \log t = F(\hat{u})\, d\hat{u} \quad , \tag{14}$$

where,

$$F(\hat{u}) = \frac{1}{\hat{u} - 2/(2 + n)} \; \frac{d \log y}{d\hat{u}} \tag{15}$$

is a rational function of $\hat{u}$ and Eq. (14) can be integrated in terms of elementary functions. The result is

$$\hat{t} = \alpha\hat{u}\, (A_2 - \hat{u})^{\beta_1} \cdot (\hat{u} - A_2/\gamma)^{\beta_2} \cdot (\hat{u} - A_5^{-1})^{\beta_3} \quad , \tag{16}$$

where

$$\alpha = A_1(A_2 - A_1^{-1})^{-\beta_1}\, (A_1^{-1} - A_2/\gamma)^{-\beta_2}\, (A_1^{-1} - A_5^{-1})^{-\beta_3} \quad ,$$

$$\beta_1 = (2\gamma + n - 2)[n(\gamma - 2)]^{-1} \cdot \beta_2 \quad ,$$

$$\beta_2 = n\gamma(2 + n)(2 - \gamma)(\gamma - 1)\, [2(2n\gamma + 2\gamma - n + 2)(2 - n - 2\gamma) +$$

$$+ 2n\gamma(2 - \gamma)(n\gamma + 4) + 2(\gamma + 1)(2 + n)(n\gamma^2 - 2n\gamma + n + 2\gamma - 2)]^{-1}$$

$$\beta_3 = (2 + n)(n\gamma^2 - 2n\gamma + n + 2\gamma - 2)[n(2 - \gamma)(n\gamma - n + 2)]^{-1} \cdot \beta_2 - 1 \quad , \tag{17}$$

and $\hat{t} = t/t_o$ is a dimensionless time normalized with respect to some convenient time scale t_o.

To complete the solution for the particle lines, an expression among $(r, t, \hat{u})$ is obtained from Eqs. (2) and (6). In dimensionless form, this expression becomes,

$$\hat{r} = (A_1\hat{u})^{-A_2} [A_3(\gamma\hat{u}/A_2 - 1)]^{A_8} [A_1(1 - A_5\hat{u})/(A_1 - A_5)]^{A_6-A_9} \hat{t}^{2/(2+n)} \quad , \tag{18}$$

where

$$\hat{r} = K_n^{-1} (\rho_o/E_o t_o^{\,2})^{1/(2+n)} r \quad . \tag{19}$$

Equations (16)-(19) form the closed form parametric solution for the family of (u)-characteristics or particle lines behind a self-similar blast wave.

IV. <u>(u ± a)-Characteristics</u>

According to the definitions of the (u ± a)-characteristics, we have

$$\frac{d_\pm r}{dt} = u \pm a \quad . \tag{20}$$

Thus, from the self-similar solution, Eqs. (1)-(7), it may be demonstrated that,

$$\frac{d_\pm r}{dt} = u'f \pm [\gamma p'g/(\rho'h)]^{1/2}$$

$$= [U/(\gamma+1)] \cdot \{2A_1\hat{u}y(\hat{u}) \pm [2\gamma(\gamma-1)\, g(\hat{u})/h(\hat{u})]^{1/2}\} \quad , \tag{21}$$

where $g(\hat{u})$, $h(\hat{u})$, $y(\hat{u})$ are given by Eqs. (1) and (2), and the shock speed U by Eq. (10).

Therefore, from Eqs. (1), (2), (10), (11), and (21), and after considerable manipulation, it is found that,

$$d \log t = G(\hat{u})\, d\hat{u} \quad , \tag{22}$$

where,

$$G(\hat{u}) = (d \log y/d\hat{u}) \cdot \{[\hat{u} - 2/(2+n)] \pm$$

$$[\gamma(\gamma-1)(2 - 2\hat{u} - n\hat{u})/2(2\gamma\hat{u} + n\gamma\hat{u} - 2)]^{1/2} \cdot \hat{u}\}^{-1} \tag{23}$$

The expression $d \log y/d\hat{u}$ in $G(\hat{u})$ is known and given by Eq. (13). Therefore, $G(\hat{u})$ may be reduced to an algebraic function involving square roots of second degree polynomials of $\hat{u}$ as radicals. This means that Eqs. (22) can be integrated in terms of elementary, albeit transcendental, functions. The results are:

$$\log \hat{t} = K_{\pm} + \log |A_5\hat{u}/(1 - A_5\hat{u})| \pm$$

$$(m/\ell^{1/2}) \sin^{-1} \{[\ell/(A_5^{-1} - \hat{u}) - k]/(k^2 - \ell)^{1/2}\} \quad , \tag{24}$$

where,

$$k = A_5^{-1} - (\gamma+1)[\gamma(2+n)]^{-1} \quad ,$$

$$\ell = A_5^{-2} - 2A_5^{-1} (\gamma+1)[\gamma(2+n)]^{-1} + 4[\gamma(2+n)^2]^{-1} \quad ,$$

$$m = -A_5^{-1} [(\gamma-1)/2]^{1/2} \quad . \tag{25}$$

Equations (24) and the parametric expression for $(\hat{r}, \hat{t}, \hat{u})$ given by Eq. (18) form the closed form parametric solutions for the two families of $(u \pm a)$-characteristics in the dimensionless $\hat{r}$-$\hat{t}$ plane. The constant $K_\pm$ in Eqs. (24) must be evaluated for each characteristic from a set of known values of $(\hat{r}, \hat{t}, \hat{u})$.

V. Calculational Results

Table I contains the values of the constants $A_1, A_2, \ldots, A_9$ of Eqs. (4), $\alpha, \beta_1, \beta_2, \beta_3$ of Eqs. (17), and k, ℓ, m of Eqs. (25), as well as the values of K_n as determined by Eq. (7) for $n = 1,2,3$ and $\gamma = 5/3$[†].

Figure 1 displays a typical set of solution curves expressing the dimensionless time $\hat{t}$ as functions of the dimensionless distance $\hat{r}$ for n=3 and γ=5/3. The terminating curve A is the path of the front of the blast wave. Behind A, there are three families of characteristics: (1) the solid curves are the (u)-characteristics or particle lines of Sec. III; (2) the dashed curves are the (u+a)-characteristics; and (3) the dot-dashed curves are the (u-a)-characteristics. As expected, if one's attention is fixed upon a particular point where curves of all three families intersect (such as the point labelled B in Fig. 1), one recognizes the familiar fact[††] that the angle between the (u+a)- and (u-a)-characteristics is bisected by the particle line.

[†]The evaluation of these constants is quite straightforward with the exception of K_n which requires the calculation of integrals whose integrands contain poles at the lower limits of the ranges of integration. The authors shall be glad to supply the values of K_n as well as other constants for other values of γ.

[††]See, e.g., Courant & Friedrichs[5].

Appendix

Group Theoretical Discussion Leading to the Self-Similar Solution

The basic differential equations describing the nonisentropic (particle isentropic) flow of an ideal gas behind a blast wave may be written as:

$$\phi_i(u_r, u_t, u; p_r, p_t, p; \rho_r, \rho_t, \rho; r, t) = 0 \quad ,$$

$$(i = 1, 2, 3) \quad , \qquad \text{(A1)}$$

where

$$\phi_1 \triangleq \rho u_r + u\rho_r + \rho_t + (n-1)\rho u/r \quad , \qquad \text{(A2a)}$$

$$\phi_2 \triangleq \rho u u_r + \rho u_t + p_r \quad , \qquad \text{(A2b)}$$

$$\phi_3 \triangleq \rho(u p_r + p_t) - \gamma p(u\rho_r + \rho_t) \quad , \qquad \text{(A2c)}$$

and subscripts denote partial differentiation.

Consider a one-parameter continuous group of transformations defined by

$$(\bar{r}, \bar{t}) = (A^a r, A^{-b} t) \quad , \qquad \text{(A3a)}$$

$$(\bar{u}, \bar{p}, \bar{\rho}) = (A^c u, A^d p, A^e \rho) \quad , \qquad \text{(A3b)}$$

and therefore,

$$(\bar{u}_{\bar{r}}, \bar{u}_{\bar{t}}, \bar{p}_{\bar{r}}, \bar{p}_{\bar{t}}, \bar{\rho}_{\bar{r}}, \bar{\rho}_{\bar{t}})$$

$$= (A^{c-a}u_r, A^{c+b}u_t, A^{d-a}p_r, A^{d+b}p_t, A^{e-a}\rho_r, A^{e+b}\rho_t) \quad , \tag{A4}$$

where A is the only parameter, and (a, b, c, d, e) are constants. The requirement of invariance of the basic differential equations (A1) and (A2) under this group will determine the choice of these constants. It can be shown that the transformed functions

$$\bar{\Phi}_i \triangleq \phi_i(\bar{u}_{\bar{r}}, \bar{u}_{\bar{t}}, \bar{u}; p_{\bar{r}}, \bar{p}_{\bar{t}}, \bar{p}; \bar{\rho}_{\bar{r}}, \bar{\rho}_{\bar{t}}, \bar{\rho}; \bar{r}, \bar{t}) \quad ,$$

$$(i = 1, 2, 3) \quad , \tag{A5}$$

will be proportional to the original ϕ_i's if we require that

$$c = a+b \quad , \quad e = d - 2(a+b) \quad . \tag{A6}$$

The result is:

$$(\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3) = (A^{-2a-b+d}\phi_1, A^{d-a}\phi_2, A^{-2a-b+2d}\phi_3) \quad . \tag{A7}$$

Thus, the ϕ_i's are conformal invariants and the basic differential equations, (A1) and (A2), become invariant under the group defined by Eqs. (A3), (A4), and (A6).

According to a theorem by Morgan[6], if a system of $M(\geq 1)$ partial differential equations with M dependent variables and $N(\geq 2)$ independent variables is invariant under a one-parameter continuous group of transformations of the dependent and independent variables, then the "invariant" solutions of this system of equations under the group (enlarged to include the transformation of the partial derivatives) can be expressed in terms of the solutions of a system of M differential equations with M dependent variables (F_i, i=1, ..., M) and (N-1) independent variables (ξ_j, j=1, ..., N-1). In the above, the F_i's are absolute invariants under the transformation group and ξ_j's are the (N-1) functionally independent invariants of the subgroup of transformations of the original independent variables. In the present case, M=3 and N=2; thus, it is expected that there exists a class of solutions to Eqs. (A1) and (A2) in terms of three functions, $F_i(\xi)$, (i = 1, 2, 3), of an absolute invariant ξ of the transformation group defined by Eq. (A3a). The F_i's, at least in class C^1, are invariants under Eqs. (A3) and (A6). It is straightforward to verify that,

$$\xi = rt^{a/b} \quad , \tag{A8}$$

and

$$\begin{aligned} F_1(\xi) &= r^{a_1} t^{(aa_1+a+b)/b} \cdot u \quad , \\ F_2(\xi) &= r^{a_2} t^{(aa_2+d)/b} \cdot p \quad , \\ F_3(\xi) &= r^{a_3} t^{(aa_3+d-2a-2b)/b} \cdot \rho \end{aligned} \tag{A9}$$

are such invariants. The values of (a_1, a_2, a_3) are arbitrary constants. Without loss of generality, the F_i's in (A9) has been chosen such that each contains only one of the original dependent variables, (u, p, ρ).

From Eqs. (A8) and (A9), it is concluded that there exists a class of self-similar solutions to Eqs. (A1) and (A2) of the form:

$$u = r^{-a_1} t^{-(aa_1+a+b)/b} \cdot F_1(\xi) \quad ,$$

$$p = r^{-a_2} t^{-(aa_2+d)/b} \cdot F_2(\xi) \quad , \tag{A10}$$

$$\rho = r^{-a_3} t^{-(aa_3+d-2a-2b)/b} \cdot F_3(\xi) \quad ,$$

where ξ is given by Eq. (A8). For a blast wave with zero ambient pressure, the energy integral given by Eq. (7) is a constant. This condition can be satisfied by the self-similar solution, (A10), if and only if:

(1) ξ = constant at the shock, and

(2) $a/b = -2/(2+n)$. (A11)

Thus, at the shock (i.e., at r=R),

$$\xi = Rt^{-2/(2+n)} = C \quad , \tag{A12}$$

where C is a constant. Therefore, for a self-similar blast wave,

$$R = C\, t^{2/(2+n)} \quad . \tag{A13}$$

The similarity parameter ξ may be replaced by a dimensionless parameter y defined as follows:

$$y = r/R = C^{-1} r t^{-2/(2+n)} \quad , \tag{A14}$$

where,

$$C = K_n (E_o/\rho_o)^{1/(2+n)} \quad , \tag{A15}$$

and K_n is a dimensionless constant.

The above completes the general formulation of the self-similar, blast wave solution. It is noted that there remain five free constants, (a, a_1, a_2, a_3, d), which may be assigned arbitrary values. For the special choice of

$$(a, a_1, a_2, a_3, d) = (2, -1, -2, 0, -2n) \quad , \tag{A16}$$

the similarity transformation defined by Eqs. (A9) in terms of the new parameter y becomes:

$$\begin{aligned} F_1(y) &= ut/r \quad , \\ F_2(y) &= pt^2/r^2 \quad , \\ F_3(y) &= \rho \quad . \end{aligned} \tag{A17}$$

These are essentially the transformations used in Refs. 3 and 4. While values of (a, a_1, a_2, a_3, d) other than those of (A16) may be chosen, it can be demonstrated that Eqs. (A9) cannot be made any simpler than those of (A17).

It should be noted that if the perfect gas assumption in ϕ_3 of (A2c) is replaced by one containing ionization and dissociation, or if the boundary conditions of (5) for zero ambient pressure is replaced by those for finite ambient pressure (i.e., for shock of finite strength), the group property leading to the self-similar solution such as those given by (A17) will be lost. Nevertheless, it should always be challenging to investigate problems in mathematical physics which admit invariant self-similar solutions under continuous groups of transformations, particularly when the solutions may be expressed in closed forms such as those described in this paper.

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Table I

Values of Pertinent Constants for n=3, $\gamma=5/3$

n	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
1	2.000	0.667	4.000	-5.000	1.333	2.167	0.429	0.286	2.786
2	2.667	0.500	4.000	-5.000	1.667	2.500	0.600	0.200	3.000
3	3.333	0.400	4.000	-5.000	2.000	2.733	0.692	0.154	3.154

n	α	β_1	β_2	β_3	k	ℓ	m	K_n
1	3.920	7.500	-1.071	-7.428	0.217	0.029	-0.433	1.184
2	3.779	5.000	-1.000	-5.000	0.200	0.030	-0.346	1.154
3	3.762	4.166	-0.961	-4.205	0.180	0.026	-0.289	1.152

Caption of Figure

Figure 1 Curved Characteristics Behind a Strong Blast Wave for $n=3$, $\gamma=5/3$.

Figure 1
Laporte and Chang